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A Theorem on the Representations of the Weyl Groups of Type D_n and B_{n-1}

TAKESHI TOKUYAMA

Department of Mathematics, University of Tokyo, Hongo, Tokyo, 113 Japan

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INTRODUCTION

The representation theory of the classical Weyl groups was constructed by A. Young, H. Weyl, W. Specht, and others, which is related with the partition symbols beautifully. In this paper we shall study the decomposition rule of the restricted representations of the Weyl group of type D_n into its subgroup H, called the folding subgroup of type B_{n-1}. H is a Weyl group of type B_{n-1} and is naturally embedded in the Weyl group of type D_n.

1. PRELIMINARY

1.1.

DEFINITION. The Weyl group of type B_n is a group generated by $S=\{s_1, s_2,..., s_{n-1}, t\}$ with relations

$$\begin{aligned} s_i^2=t^2&=1 \qquad \text{for} \quad i=1,2,..., n-1 \\ (s_i s_{i+1})^3&=1 \qquad \text{for} \quad i=1,2,..., n-2 \\ (s_i s_j)^2&=1 \qquad \text{for} \quad i<j, j\neq i+1 \\ (s_i t)^2&=1 \qquad \text{for} \quad i=1,2,..., n-2 \\ (s_{n-1}t)^4&=1. \end{aligned}$$

The Weyl group of type D_n is a group generated by $S_0=\{s_1, s_2,..., s_{n-2}, a, b\}$ with relations

$$\begin{aligned} s_i^2=a^2=b^2&=1 \qquad \text{for} \quad i=1,2,..., n-2 \\ (s_i s_{i+1})^3&=1 \qquad \text{for} \quad i=1,2,..., n-3 \end{aligned}$$

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$$(s_i s_j)^2 = 1 \qquad \text{for} \quad i < j, j \neq i+1$$
$$(s_i a)^2 = (s_i b)^2 = (ab)^2 = 1 \qquad \text{for} \quad i \neq n-2$$
$$(s_{n-2} a)^3 = (s_{n-2} b)^3 = 1.$$

We denote the Weyl group B_n and D_n by $\mathfrak{B}_n$ and $\mathfrak{D}_n$, respectively.

Remark. We can consider $\mathfrak{D}_n$ as a subgroup of $\mathfrak{B}_n$ regarding that $a = s_{n-1}$ and $b = ts_{n-1}t$.

For all positive integers $m < n$, $\mathfrak{B}_n$ has a subgroup $\mathfrak{B}_{m,n-m}$ generated by $\{s_1, s_2, \ldots, s_{m-1}, s_{m+1}, \ldots, s_{n-1}, t, t'\}$ where $t' = (s_1 s_2 \cdots s_{n-2} s_{n-1}) t(s_{n-1} s_{n-2} \cdots s_2 s_1)$. It is isomorphic to $B_m \times B_{n-m}$.

The subgroup of $\mathfrak{B}_n$ generated by $\{s_1, s_2, \ldots, s_{n-1}\}$ is isomorphic to $\mathfrak{S}_n$, the symmetric group of degree n.

1.2. Let us recall the construction of irreducible representations of $\mathfrak{B}_n$ and $\mathfrak{D}_n$.

DEFINITION. $P(n) := \{\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^n) \mid \lambda^1 \geqslant \lambda^2 \geqslant \cdots \geqslant \lambda^n \geqslant 0, |\lambda| = n\}$ is the set of partitions of n, where $|\lambda| = \sum_{i=1}^n \lambda^i$. We define $P(0) := \{\phi\}$.

$BP(n) := \{(\lambda, \mu) \mid \lambda$ and μ are partitions such that $|\lambda| + |\mu| = n\}$.

It is well known that the isomorphic classes of irreducible representations of $\mathfrak{S}_n$ are in one-to-one correspondence with the partitions of n (cf. [1]). Similarly, those of $\mathfrak{B}_n$ are in one-to-one correspondence with the elements of $BP(n)$ as follows.

Let ρ_λ be the representation of $\mathfrak{S}_n$ corresponding to the partition λ, and V_λ be its representation space. Using the subgroup $\mathfrak{S}_n$ of $\mathfrak{B}_n$ defined in 1.1, we shall define representations $\rho(\lambda, \phi)$ and $\rho(\phi, \lambda)$ as below.

(1) $\rho(\lambda, \phi)(s_i) = \rho_\lambda(s_i)$ for $i = 1, 2, \ldots, n-1$.
$\rho(\lambda, \phi)(t) = I_{V_\lambda}$ (identity transformation of V_λ).

(2) $\rho(\phi, \lambda)(s_i) = \rho_\lambda(s_i)$ for $i = 1, 2, \ldots, n-1$.
$\rho(\phi, \lambda)(t) = -I_{V_\lambda}$.

These representations are clearly well defined. Then we define a representation $\rho(\lambda, \mu)$ for an element (λ, μ) of $BP(n)$ as the induced representation of $\rho(\lambda, \phi) \times \rho(\phi, \mu)$ from $\mathfrak{B}_{|\lambda|, |\mu|}$ to $\mathfrak{B}_n$.

Then we get the following theorem.

THEOREM (cf. [3]). *$\{\rho(\lambda, \mu) \mid (\lambda, \mu) \in BP(n)\}$ is a complete representative system of the irreducible representations of $\mathfrak{B}_n$.*

As an abbreviation we shall simply write (λ, μ) for $\rho(\lambda, \mu)$.

Since $\mathfrak{D}_n$ is a subgroup of $\mathfrak{B}_n$ of index 2 and it is the kernel of the representation $(\phi, (n))$ indeed, we shall construct its irreducible representations using the restricted representations of those of $\mathfrak{B}_n$.

DEFINITION. Let (λ, μ) be an irreducible representation of $\mathfrak{B}_n$. Then we define a representation $[\lambda, \mu]$ of $\mathfrak{D}_n$ by

$$[\lambda, \mu] = (\lambda, \mu)|_{\mathfrak{D}_n} \qquad \text{(restricted representation)}.$$

PROPOSITION (cf. [5]). (1) *$[\lambda, \mu]$ is irreducible if and only if $\lambda \neq \mu$.*

(2) *$[\lambda, \lambda]$ is decomposed into two irreducible constituents $\langle\lambda\rangle_+$ and $\langle\lambda\rangle_-$.*

(3) *$[\lambda, \mu] = [\lambda', \mu']$ if and only if $\lambda = \lambda'$ and $\mu = \mu'$, or $\lambda = \mu'$ and $\mu = \lambda'$.*

(4) *$\{[\lambda, \mu] \mid \lambda > \mu\} \cup \{\langle\lambda\rangle_+, \langle\lambda\rangle_- \mid |\lambda| = n/2\}$ is a complete representative sytem of the irreducible representations of $\mathfrak{D}_n$; here $>$ is the Bruhat order.*

2.

2.1. Let W be a Weyl group of type D_n generated by $S = \{s_1, s_2, \ldots, s_{n-1}, a, b\}$ as in 1.1. We shall consider its subgroup H generated by $\{s_1, s_2, \ldots, s_{n-2}, ab\}$. H is a Weyl group of type B_{n-1} naturally and called the folding subgroup of type B_{n-1} in $\mathfrak{D}_n$.

Our theorem is concerned with the relation of the representations of W and H.

DEFINITION. Let λ and μ be partitions. We define a relation $\rhd$ so that $\lambda \rhd \mu$ if and only if there exists an i such that $\lambda^i = \mu^i + 1$ and $\lambda^j = \mu^j$ for all other j. $\lambda \lhd \mu$ means $\mu \rhd \lambda$.

Our theorem is the following.

THEOREM. *Let $[\lambda, \mu]$ be a representation of W. Then its restricted representation of H is decomposed into the irreducible constituents as shown below.*

$$[\lambda, \mu]|_H = \sum_{\alpha \lhd \lambda} (\alpha, \mu) + \sum_{\beta \lhd \mu} (\beta, \lambda).$$

COROLLARY. *Let $\langle\lambda\rangle_+$ be an irreducible representation of W. Then,*

$$\langle\lambda\rangle_+|_H = \sum_{\alpha \lhd \lambda} (\alpha, \lambda).$$

Corollary follows the theorem immediately since $\langle\lambda\rangle_+|_H = \langle\lambda\rangle_-|_H$.

Proof of the theorem. Let W_0 be a Weyl group of type B_n generated by $S_0 = \{s_1, s_2, ..., s_{n-1}, t\}$ as in 1.1. Regarding that $a = s_{n-1}$ and $b = s_{n-1}ts_{n-1}$, we consider W and H as subgroups of W_0. Let W_1 be the subgroup of W_0 generated by $S_1 = \{s_1, s_2, ..., s_{n-2}, t'\}$ where $t' = y^{-1}ty$ and $y = s_{n-2} \cdot s_{n-3} \cdots s_1$. W_1 is a Weyl group of type B_{n-1}. Let T be the cyclic group of order 2 generated by t and W_2 be the direct product of W_1 and T in W_0 (which is a subgroup isomorphic to $\mathfrak{B}_{n-1,1}$). Then H is a subgroup of W_2. To distinguish the representations of W_1 from those of H, these groups are both Weyl groups of type B_{n-1}, we shall denote $(\lambda, \mu)_1$ and $(\lambda, \mu)_h$ for the representation of W_1 and H, respectively, which are equivalent to (λ, μ) as a representation of $\mathfrak{B}_{n-1}$.

Let $[\lambda, \mu]$ be a representation of W. By definition $[\lambda, \mu] = (\lambda, \mu)|_W^{W_0}$ (the restricted representation of the representation (λ, μ) of W_0 into W). So $[\lambda, \mu]|_H = (\lambda, \mu)|_H$.

We shall prepare the following lemma, which is a special case of Corollary 2.6 in [6].

LEMMA. *Let (λ, μ) be an irreducible representation of W_0. Then*

$$(\lambda, \mu)|_{W_2} = \sum_{\alpha \lhd \lambda} (\alpha, \mu)_1 \times ((1), \phi) + \sum_{\beta \lhd \mu} (\lambda, \beta)_1 \times (\phi, (1)).$$

Comparing this lemma with the theorem, it now suffices to show the following claim.

Claim. Let (ν, τ) be an irreducible representation of B_{n-1}. Then

(1) $(\nu, \tau)_1 \times ((1), \phi) \downarrow_H^{W_2} = (\nu, \tau)_h$,

(2) $(\nu, \tau)_1 \times (\phi, (1)) \downarrow_H^{W_2} = (\tau, \nu)_h$.

Proof. We denote the character value of a representation ρ at an element g by $X\{\rho\}(g)$. Then we have the following from the definition (see 1.2).

$$\begin{aligned} X\{(\nu, \tau)_1 \times (\phi, (1))\}(s_i) &= X\{(\nu, \tau)_h\}(s_i) \\ &= X\{(\tau, \nu)_h\}(s_i) \end{aligned} \tag{2.1.1}$$

for all $i = 1, 2, ..., n-2$.

$$\begin{aligned} X\{(\nu, \tau)_1 \times (\phi, (1))\}(ab) &= X\{(\nu, \tau)_1\}(s_{n-2}ts_{n-2}) \times X\{(\phi, (1))\}(t) \\ &= -X\{(\nu, \tau)_1\}(t') \\ &= -X\{(\nu, \tau)_h\}(ab) \\ &= X\{(\tau, \nu)_h\}(ab). \end{aligned} \tag{2.1.2}$$

Equations (2.1.1) and (2.1.2) indicate (2). We can check (1) similarly.

Q.E.D.

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